JOURNAL OF NUMBER THEORY 4, 274–285 (1972)

An Analog to the Lagrange Numbers

BOHUSLAV DIVIŠ

Department of Mathematics, Ohio State University, Columbus, Ohio 43210

Communicated by H. Zassenhaus

Received August 12, 1970

Let β be an irrational number. For $t \geqslant 1$, put

$$\psi_\beta(t) = \min_{\substack{p,q \text{ int} \\ 0<q\leqslant t}} |q\beta - p|, \quad \mu^*(\beta) = \sup_{t \geqslant 1} t\psi_\beta(t)$$

and write $\mu^*(\beta) = R_\beta^*/(1 + R_\beta^*)$. Further, denote by $\mathscr{M}^*$ the set of all values R_β^* when β runs over all irrational numbers. The author proves that $\mathscr{M}^*$ contains a sequence of elements between $1 + \sqrt{5}$ and $2 + \sqrt{5}$ tending to $2 + \sqrt{5}$ but does not contain $2 + \sqrt{5}$ itself and that a number R^{**} exists such that the whole interval $[R^{**}, \infty] \subset \mathscr{M}^*$.

Let β be an irrational number and let $(b_0; b_1, b_2, \ldots)$ its (simple) continued fraction expansion. For real $t \geqslant 1$, let

$$\psi_\beta(t) = \min_{\substack{p,q \text{ int} \\ 0<q\leqslant t}} |q\beta - p|.$$

It is well known that $0 < t\psi_\beta(t) < 1$ for every $t \geqslant 1$.
Let

$$\mu(\beta) = \limsup_{t \to +\infty} t\psi_\beta(t), \qquad \mu^*(\beta) = \sup_{t \geqslant 1} t\psi_\beta(t).$$

The purpose of this paper is to prove several theorems concerning the numbers $\mu^*(\beta)$.

Firstly we prove the following simple lemma.

LEMMA 1. *For each irrational number β we have*

$$\mu^*(\beta) = \cfrac{1}{1 + \cfrac{1}{R_\beta^*}},$$

where

$$R_\beta^* = \sup_{k\geqslant 1} (b_k ; b_{k-1}, \ldots, b_1) \cdot (b_{k+1} ; b_{k+2}, \ldots) \quad \left(\frac{1}{R_\beta^*} = 0 \textit{ for } R_\beta^* = +\infty\right).$$

It is sufficient to prove the lemma for $0 < \beta < 1$. If p_n/q_n denotes the n-th convergent of β, then obviously

$$\mu^*(\beta) = \sup_{k\geqslant 1} q_{k+1} \mid q_k\beta - p_k \mid.$$

Now (see, e.g., [1, Chap. 1, Section 2)

$$q_{k+1} \mid q_k\beta - p_k \mid = (1 + \theta_{k+1}\varphi_k)^{-1},$$

where $\theta_{k+1} = (0; b_{k+1}, b_{k+2}, \ldots)$, $\varphi_k = q_k/q_{k+1} = (0; b_k, b_{k-1}, \ldots, b_1)$. From the lemma it follows immediately that $\frac{1}{2} \leqslant \mu^*(\beta) \leqslant 1$ $(1 \leqslant R_\beta^* \leqslant +\infty)$ and we have $\mu^*(\beta) < 1$ $(R_\beta^* < +\infty)$ if and only if the sequence $b_1, b_2, b_3, \ldots$ is bounded.

Since the numerical expressions for the numbers $\mu^*(\beta)$ would be too complicated we shall formulate our results in terms of R_β^*.

Throughout this paper we shall use the following notations. A number $\beta' = (b_0'; b_1', b_2', \ldots)$ will be called equivalent to β if there exists an integer n such that $b'_{k+n} = b_k$ for all sufficiently large k. We shall use the notation $\beta' \sim \beta$ or $\beta' \nsim \beta$ according to whether β' and β are equivalent or not. A number $\beta'' = (b_0''; b_1'', b_2'', \ldots)$ will be called strongly equivalent to β if one of these conditions is satisfied:

(1) $b_j = b_j''$, for $j \geqslant 1$,
(2) $b_1 \geqslant 2$, $b_1'' = 1$, $b_2'' = b_1 - 1$, and $b''_{j+1} = b_j$, for $j \geqslant 2$,
(3) $b_1'' \geqslant 2$, $b_1 = 1$, $b_2 = b_1'' - 1$, and $b_{j+1} = b_j''$, for $j \geqslant 2$.

We shall use the notation $\beta'' \approx \beta$ or $\beta'' \not\approx \beta$ according to whether β'' and β are strongly equivalent or not. If $\beta'' \approx \beta$ then obviously $R_{\beta''}^* = R_\beta^*$ i.e., $\mu^*(\beta'') = \mu^*(\beta)$. We shall use a standard notation for the period of a continued fraction; e.g., $(1; \overline{1, 2}) = (1; 1, 2, 1, 2, \ldots) = \sqrt{3}$. By a_n $(n \geqslant 0)$ we denote the n-th term of the Fibonacci sequence

$$(a_0 = a_1 = 1,\ a_{n+2} = a_{n+1} + a_n \text{ for } n \geqslant 0).$$

We put $\mu(\beta) = (1/(1 + 1/R_\beta))$ and let $\mathscr{M} = \bigcup_\beta \{R_\beta\}$. Similarly, let $\mathscr{M}^* = \bigcup_\beta \{R_\beta^*\}$. Some basic properties of the set $\mathscr{M}$ were studied already in [2, 3]. Our Theorem 1 (compare with [2] or [3, Theorem 1] and Theorem 3 (compare with [3, Theorem 4]) express certain similarities between the sets $\mathscr{M}$ and $\mathscr{M}^*$. On the other hand, Theorem 2 shows there is a significant difference between them.

THEOREM 1. *Let* $c_i = 1$ $(i = 0, 1, 2,...)$, $\alpha_0 = (c_0; c_1, c_2, ...)$,

$$\alpha_n = \overline{(2; c_1, c_2, ..., c_{2n-1})} \qquad \textit{for} \quad n = 1, 2, ...$$

$$\alpha_n' = (c_0\, ; \overline{2, c_1, c_2, ..., c_{2n-1}}) \qquad \textit{for} \quad n = 1, 2, ...$$

Then we have

(a) $R^*_{\alpha_0} = 1 + \sqrt{5}$,

(b) $R^*_{\alpha_n} = R^*_{\alpha_n'} = \dfrac{a_{2n+1}}{a_{2n}} \left(1 + \sqrt{\dfrac{a_{2n+1}}{a_{2n-1}}}\right)$, *for* $n = 1, 2, ...,$

(c) $R^*_{\alpha_n} < R^*_{\alpha_{n+1}}$, *for* $n = 0, 1, 2, ...,$

(d) $\lim\limits_{n\to+\infty} R^*_{\alpha_n} = 2 + \sqrt{5}$,

(e) *If* $R_\beta^* < 2 + \sqrt{5}$, *then either* $\beta \approx \alpha_i$ *for some* $i \geqslant 0$ *or* $\beta \approx \alpha_j'$ *for some* $j \geqslant 1$.

Proof. The statements (a), (b), and (d) can be verified by a direct calculation; (a) and (d) are very easy; (b) is based on the relation

$$\alpha_n = 1 + \sqrt{\frac{a_{2n+1}}{a_{2n-1}}} \qquad (n = 1, 2, ...).$$

Proof of (c). Let us write

$$R^*_{\alpha_n} = \frac{a_{2n+1}}{a_{2n}} \left(1 + \sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right) = \frac{1 + \sqrt{\dfrac{a_{2n+1}}{a_{2n-1}}}}{1 - \dfrac{a_{2n-1}}{a_{2n+1}}}$$

and let us put $f_1(x) = (1 + x)/(1 - 1/x^2)$ for $x > 1$. We easily see that

$$f_1\left(\sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right) = R^*_{\alpha_n} \qquad (n = 1, 2, ...).$$

Since

$$3 = \frac{a_3}{a_1} > \frac{a_5}{a_3} > \cdots > \left(\frac{1 + \sqrt{5}}{5}\right)^2,$$

the statement (c) will be proved when we show that $f_1(x)$ is a decreasing function for $(1 + \sqrt{5})/2 \leqslant x \leqslant \sqrt{3}$. However,

$$f_1(x) = \frac{1+x}{1-\dfrac{1}{x^2}} = x + 1 + \frac{1}{x-1}$$

and thus $f_1'(x) = 1 - (1/(x-1)^2)$, so that $f_1'(x) < 0$ for $1 < x < 2$.

Proof of (e). Here lies the kernel of the theorem. It was proved by Lesca [2, 3], that if $R_\beta < 2 + \sqrt{5}$, then $\beta \sim \alpha_n$ for some $n \geqslant 0$. Since, obviously, $R_\beta{}^* \geqslant R_\beta$ for every β, we have the following simple consequence: If $R_\beta{}^* < 2 + \sqrt{5}$, then $\beta \sim \alpha_m$ for some $m \geqslant 0$.

1. First we show the following:

$$\beta \sim \alpha_0 \;\&\; \beta \not\approx \alpha_0 \Rightarrow R_\beta{}^* > 2 + \sqrt{5}.$$

For, if we had $\beta \sim \alpha_0$, $\beta \not\approx \alpha_0$ and $R_\beta{}^* \leqslant 2 + \sqrt{5}$, then

$$\text{either } \beta = (b_0; b_1, ..., b_k, \bar{1}), \text{ where } k \geqslant 2,\ b_k \geqslant 2 \tag{1}$$
$$\text{or } \beta = (b_0; b_1, \bar{1}), \qquad \text{where } b_1 \geqslant 3. \tag{2}$$

In the case (1) we obviously must have $b_k = 2$, since otherwise

$$R_\beta{}^* \geqslant (b_k; b_{k-1}, ..., b_1) \cdot (\bar{1}) \geqslant \tfrac{3}{2}(1 + \sqrt{5}) > 2 + \sqrt{5}.$$

Further, we must have:

$$2 + \sqrt{5} \geqslant R_\beta{}^* \geqslant (b_{k-1}; b_{k-2}, ..., b_1) \cdot (2; \bar{1}),$$
$$2 + \sqrt{5} \geqslant R_\beta{}^* \geqslant (2; b_{k-1}, ..., b_1) \cdot (\bar{1}),$$

and hence

$$(b_{k-1}; b_{k-2}, ..., b_1) < \frac{2+\sqrt{5}}{(2;\bar{1})} = \frac{1+\sqrt{5}}{2} = (\bar{1}),$$

$(2; b_{k-1}, ..., b_1) < (2 + \sqrt{5})/(\bar{1}) = (2; \bar{1})$, which is a contradiction. In the case (2) we have $R_\beta{}^* \geqslant 3.\ (\bar{1}) > 2 + \sqrt{5}$.

2. Let $n \geqslant 1$, $\beta \sim \alpha_n$, $\beta \not\approx \alpha_n{}'$ and $R_\beta{}^* \leqslant 2 + \sqrt{5}$. We write $\beta = (b_0; b_1, b_2, ...)$ and denote by k the least positive suffix such that

$$b_{k+1} = 2,\ b_j = 1\ (k + 2 \leqslant j \leqslant k + 2n),\ b_{i+2n} = b_i\ (i \geqslant k + 1).$$

Thus we have $\beta = (b_0; b_1, ..., b_k, \overline{2, c_1, c_2, ..., c_{2n-1}})$, $k \geqslant 1$ (we are using again the notation $c_j = 1, j \geqslant 1$).

By our hypothesis, we have

$$2+\sqrt{5} \geqslant R_\beta^* \geqslant (b_k\,;b_{k-1}\,,b_{k-2}\,,\dots,b_1)\cdot\overline{(2;c_1\,,c_2\,,\dots,c_{2n-1})}, \tag{3}$$

$$2+\sqrt{5} \geqslant R_\beta^* \geqslant (2;b_k\,,\dots,b_1)\cdot\overline{(c_1\,;c_2\,,\dots,c_{2n-1}\,,2)}. \tag{4}$$

With respect to the relation

$$\overline{(2;c_1\,,c_2\,,\dots,c_{2n-1})} = 1+\sqrt{\frac{a_{2n+1}}{a_{2n-1}}},$$

$$\overline{(c_1\,;c_2\,,\dots,c_{2n-1}\,,2)} = \frac{a_{2n-1}}{a_{2n}}\left(1+\sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right),$$

we have, by the inequalities (3) and (4),

$$(b_k\,;b_{k-1}\,,\dots,b_1) \leqslant \frac{2+\sqrt{5}}{1+\sqrt{\dfrac{a_{2n+1}}{a_{2n-1}}}}, \tag{5}$$

$$(2;b_k\,,\dots,b_1) \leqslant \frac{2+\sqrt{5}}{1+\sqrt{\dfrac{a_{2n+1}}{a_{2n-1}}}}\,\frac{a_{2n}}{a_{2n-1}}. \tag{6}$$

3. Further, we prove the following relations:

$$\frac{2+\sqrt{5}}{1+\sqrt{\dfrac{a_{2n+1}}{a_{2n-1}}}} < \frac{a_{2n+1}+a_{2n+3}}{a_{2n}+a_{2n+2}}, \qquad n=1,2,\dots, \tag{7}$$

$$\frac{2+\sqrt{5}}{1+\sqrt{\dfrac{a_{2n+1}}{a_{2n-1}}}}\,\frac{a_{2n}}{a_{2n-1}} < \frac{6a_{2n+1}+a_{2n-1}}{6a_{2n-1}+a_{2n-3}}, \qquad n=1,2,\dots \quad (a_{-1}=0). \tag{8}$$

Proof of (7). Let us write (7) in the form

$$\left(1+\sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right)\frac{a_{2n+1}+a_{2n+3}}{a_{2n}+a_{2n+2}}$$

$$=\left(1+\sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right)\frac{4a_{2n+1}-a_{2n-1}}{3a_{2n+1}-2a_{2n-1}} > 2+\sqrt{5}.$$

Let us put $f_2(x) = (1 + x)(4x^2 - 1)/(3x^2 - 2)$. Hence, we have

$$f_2\left(\sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right) = \left(1 + \sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right) \frac{4a_{2n+1} - a_{2n-1}}{3a_{2n+1} - 2a_{2n-1}}$$

and $f_2((1 + \sqrt{5})/2) = 2 + \sqrt{5}$. Since

$$3 = \frac{a_3}{a_1} > \frac{a_5}{a_3} > \cdots > \left(\frac{1 + \sqrt{5}}{2}\right)^2,$$

for the proof of (7) it suffices to show that $f_2'(x) > 0$ for

$$\frac{1 + \sqrt{5}}{2} \leqslant x \leqslant \sqrt{3}.$$

For such x we have

$$\begin{aligned} &f_2'(x)(3x^2 - 2)^2 \\ &\quad = 12x^4 - 21x^2 - 10x + 2 > 12x^4 - 21x^2 - 10x + 2 - (2 - x)(8 - x) \\ &\quad = 12x^4 - 22x^2 - 14 = 2(2x^2 + 1)(3x^2 - 7) > 0, \end{aligned}$$

since $(1 + \sqrt{5})/2 > \sqrt{7}/\sqrt{3}$.

Proof of (8). Let us write (8) in the form

$$\begin{aligned} &\frac{6a_{2n+1} + a_{2n-1}}{6a_{2n-1} + a_{2n-3}} \frac{a_{2n-1}}{a_{2n}} \left(1 + \sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right) \\ &\quad = \frac{6a_{2n+1} + a_{2n-1}}{9a_{2n-1} - a_{2n+1}} \frac{a_{2n-1}}{a_{2n+1} - a_{2n-1}} \left(1 + \sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right) > 2 + \sqrt{5}. \end{aligned}$$

Let us put $f_3(x) = (6x^2 + 1)/(9 - x^2) \cdot (1 + x)/(x^2 - 1)$. Then we have

$$f_3\left(\sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right) = \frac{6a_{2n+1} + a_{2n-1}}{9a_{2n-1} - a_{2n+1}} \frac{a_{2n-1}}{a_{2n+1} - a_{2n-1}} \left(1 + \sqrt{\frac{a_{2n+1}}{a_{2n-1}}}\right)$$

and $f_3((1 + \sqrt{5})/2) = 2 + \sqrt{5}$. Since

$$3 = \frac{a_3}{a_1} > \frac{a_5}{a_3} > \cdots > \left(\frac{1 + \sqrt{5}}{2}\right)^2,$$

it suffices for the proof of (8) to show that $f_3'(x) > 0$ for

$$\frac{1 + \sqrt{5}}{2} \leqslant x \leqslant \sqrt{3}.$$

For such x we have

$$
\begin{aligned}
& f_3'(x)(9-x^2)^2(x-1)^2 \\
& \quad = 6x^4 + 57x^2 - 110x - 9 \geqslant 6x^4 + 57x^2 - 110x - 9 - 4(3x-5)^2 \\
& \quad = 6x^4 + 21x^2 + 10x - 109 \\
& \quad \geqslant 6\left(\frac{1+\sqrt{5}}{2}\right)^4 + 21\left(\frac{1+\sqrt{5}}{2}\right)^2 + 10\left(\frac{1+\sqrt{5}}{2}\right) - 109 \\
& \quad = \tfrac{1}{2}(49\sqrt{5} - 103) > 0.
\end{aligned}
$$

4. By (5) and (7), we have

$$(b_k\,;\, b_{k-1}, \dots, b_1) < \frac{a_{2n+1} + a_{2n+3}}{a_{2n} + a_{2n+2}}, \qquad n = 1, 2, \dots \tag{9}$$

Analogously, by (6) and (8), we have

$$(2;\, b_k, b_{k-1}, \dots, b_1) < \frac{6a_{2n+1} + a_{2n-1}}{6a_{2n-1} + a_{2n-3}}, \qquad n = 1, 2, \dots \quad (a_{-1} = 0). \tag{10}$$

Now, we have, for $n \geqslant 1$

$$\frac{a_{2n+1} + a_{2n+3}}{a_{2n} + a_{2n+2}} = (c_1\,;\, c_2, \dots, c_{2n+1}, 3), \tag{11}$$

$$\frac{6a_{2n+1} + a_{2n-1}}{6a_{2n-1} + a_{2n-3}} = \begin{cases} (3;\, 6), & \text{for } n = 1 \quad (a_{-1} = 0), \\ (2;\, c_1, \dots, c_{2n-3}, 2, 6), & \text{for } n \geqslant 2. \end{cases} \tag{12}$$

The relations (11) and (12) can be verified by a direct computation. Thus, we can write, instead of (9) and (10),

$$(b_k\,;\, b_{k-1}, \dots, b_1) < (c_1\,;\, c_2, \dots, c_{2n+1}, 3), \qquad n = 1, 2, \dots, \tag{13}$$

$$(b_k\,;\, b_{k-1}, \dots, b_1) > \begin{cases} (0;\, 1, 6), & n = 1, \\ (c_1\,;\, c_2, \dots, c_{2n-3}, 2, 6), & n = 2, 3, \dots . \end{cases} \tag{14}$$

As a next step, we prove the following:

LEMMA 2. *If* $R_\beta^* \leqslant 2 + \sqrt{5}$, *where* $\beta = (b_0;\, b_1, b_2, \dots)$, *then either* $b_j \leqslant 2$, *for all* $j \geqslant 1$ *or* $b_1 = 3$, $b_{2j} = 1$, $b_{2j+1} = 2$, *for* $j \geqslant 1$.

Proof. Let $\beta = (b_0;\, b_1, b_2, \dots)$ and $R_\beta^* \leqslant 2 + \sqrt{5}$. If we had $b_l \geqslant 5$ for some $l \geqslant 1$, then we would also have $R_\beta^* \geqslant 5$. Thus, let $b_j \leqslant 4$ for all $j \geqslant 1$. If we had $b_l = 4$ for some $l \geqslant 1$, then we would have

$R_\beta^* \geqslant 4 \cdot (1; 4, 1) = 24/5 > 2 + \sqrt{5}$. Hence, we have $b_j \leqslant 3$ for all $j \geqslant 1$. If $b_l = 3$ for some $l \geqslant 2$, then

$$2 + \sqrt{5} \geqslant R_\beta^* \geqslant (b_{l-1}; b_{l-2}, \dots, b_1) \cdot 3, \tag{15}$$

$$2 + \sqrt{5} \geqslant R_\beta^* \geqslant (3; b_{l-1}, b_{l-2}, \dots, b_1) \cdot (1; 3, 1). \tag{16}$$

By (15) we have

$$(b_{l-1}; b_{l-2}, \dots, b_1) \leqslant \frac{2 + \sqrt{5}}{3} < \frac{5}{3}, \tag{17}$$

by (16) we have $(3; b_{l-1}, \dots, b_1) \leqslant \frac{4}{5}(2 + \sqrt{5}) < \frac{17}{5}$; thus

$$(b_{l-1}; b_{l-2}, \dots, b_1) > \tfrac{5}{2},$$

in contradiction with (17). Hence, it suffices to consider the case when $b_1 = 3$ and $b_j \leqslant 2$ for $j \geqslant 2$. By assumption, we must have

$$2 + \sqrt{5} \geqslant R_\beta^* \geqslant 3 \cdot (b_2; b_3, \dots),$$

hence $(b_2; b_3, \dots) \leqslant (2 + \sqrt{5})/3 < \frac{3}{2} = (1; 2)$, thus necessarily $b_2 = 1$ and $b_3 = 2$. Further, we must have $2 + \sqrt{5} \geqslant R_\beta^* \geqslant (2; 1, 3) \cdot (b_4; b_5, \dots)$, hence

$$(b_4; b_5, \dots) \leqslant \frac{2 + \sqrt{5}}{(2; 1, 3)} = \frac{4}{11}(2 + \sqrt{5}) < \frac{17}{11} = (1; 1, 1, 5),$$

thus also $b_4 = 1$ and $b_5 = 2$. Further, we must have

$$2 + \sqrt{5} \geqslant R_\beta^* \geqslant (2; 1, 2, 1, 3) \cdot (b_6; b_7, \dots)$$

hence

$$(b_6; b_7, \dots) \leqslant \frac{2 + \sqrt{5}}{(2; 1, 2, 1, 3)} < \frac{2 + \sqrt{5}}{(2; \overline{1})} = \frac{2 + \sqrt{5}}{1 + \sqrt{3}} < \frac{\frac{17}{4}}{\frac{30}{11}} < (1; 1, 1, 3); \tag{18}$$

thus necessarily $b_6 = 1$ and $b_7 = 2$. It is obvious that the argument used in the inequality (18) may be applied at each following step, thus Lemma 2 is proved.

Now, we are already in a position to finish the proof of Theorem 1. From the inequalities (13) and (14), from the definition of the suffix k, by Lemma 2, and from the relation $\beta \not\approx \alpha_n'$, it follows that we have only the following possibilities:

$$\text{for } n = 1; \quad \left.\begin{array}{ll}(1) & k = 1, b_1 = 1 \\ (2) & k = 2, b_1 = 3, b_2 = 1\end{array}\right\} \beta \approx \alpha_1$$

$$\text{for } n \geqslant 2; \quad \left.\begin{array}{ll}(1) & k = 2n - 2;\ b_1 = 2,\ b_j = 1 \ \ (2 \leqslant j \leqslant 2n - 2) \\ (2) & k = 2n - 1;\ b_j = 1 \ \ (1 \leqslant j \leqslant 2n - 1).\end{array}\right\} \beta \approx \alpha_n$$

Theorem 1 is proved.

Remark. Notice also that we have in fact proved the following:

If $\beta \sim \alpha_n$, $\beta \not\approx \alpha_n$ and $\beta \not\approx \alpha_n'$, then $R_\beta^* > 2 + \sqrt{5}$.

If $\beta \sim \alpha_0$ and $\beta \not\approx \alpha_0$, then $R_\beta^* > 2 + \sqrt{5}$.

Now, we ask: Does also the point $2 + \sqrt{5}$ belong to the set $\mathscr{M}^*$? The answer is given by next theorem.

THEOREM 2. *There exists no irrational number β with $R_\beta^* = 2 + \sqrt{5}$.*

Proof. If there were such a β, what form might it have? We write $\beta = (b_0; b_1, \ldots)$. By Lemma 2, we can have only $b_1 = 1$ or $b_1 = 2$. If $b_1 = 1$, then also $b_2 = 1$, since

$$(b_0; 1, 2, b_3, b_4, \ldots) \approx (b_0; 3, b_3, b_4, \ldots).$$

On the other hand, we have

$$(b_0; 1, 1, b_3, b_4, \ldots) \approx (b_0; 2, b_3, b_4, \ldots);$$

thus, it suffices to consider only β of the form

$$\beta = (b_0; 2, b_2, b_3, \ldots).$$

By Lemma 2, we have necessarily $b_j \leqslant 2$ for $j \geqslant 1$. We will show that we cannot have $b_k = b_{k+1} = 2$ for any $k \geqslant 1$. If this were so, then we would have

$$R_\beta^* \geqslant (b_k; b_{k-1}, \ldots, b_1) \cdot (b_{k+1}; b_{k+2}, \ldots) > 2 \cdot (2; 2, 1) = \tfrac{14}{3} > 2 + \sqrt{5},$$

which is a contradiction. Hence, we may consider only those β which have the form (remember that $c_j = 1$ for all $j \geqslant 1$)

$$\beta = (b_0\,; 2, c_1, c_2, \ldots, c_{n_1}, 2, c_1, c_2, \ldots, c_{n_2}, 2, \ldots, 2, c_1, c_2, \ldots, c_{n_k}, 2, \ldots), \tag{19}$$

where $n_k \geqslant 1$ when $k \geqslant 1$. Now, we show that all $n_k (k \geqslant 1)$ must be odd. If we had, namely, n_k even and n_{k+1} odd for some $k \geqslant 1$, then we would have

$$R_\beta{}^* \geqslant (c_1\,; c_2\,,..., c_{n_k}\,, 2) \cdot (2; c_1\,, c_2\,,..., c_{n_{k+1}}\,, 2) > (\bar{1}) \cdot (2; \bar{1}) = 2 + \sqrt{5}.$$

By the same argument, we cannot have n_k odd and n_{k+1} even for any k. It remains for us to exclude the case when n_k is even and n_{k+1} is even for some $k \geqslant 1$. Without loss of generality, let $2 \leqslant n_k \leqslant n_{k+1}$. Then

$$\begin{aligned} R_\beta{}^* &\geqslant (c_1\,; c_2\,,..., c_{n_k}\,, 2) \cdot (2; c_1\,, c_2\,,..., c_{n_{k+1}}\,, 3) \\ &\geqslant (c_1\,; c_2\,,..., c_{n_k}\,, 2) \cdot (2; c_1\,, c_2\,,..., c_{n_k}\,, 3) \\ &= \frac{a_{n_k+2}}{a_{n_k+1}}\left(1 + \frac{a_{n_k+1} + a_{n_k+3}}{a_{n_k} + a_{n_k+2}}\right) = \frac{a_{n_k+2}}{a_{n_k+1}}\left(1 + \frac{a_{n_k+2} + 2a_{n_k+1}}{2a_{n_k+2} - a_{n_k+1}}\right). \end{aligned}$$

Let us put $f_4(x) = x(1 + (x + 2)/(2x - 1))$. We have

$$f_4\left(\frac{a_{n_k+2}}{a_{n_k+1}}\right) = \frac{a_{n_k+2}}{a_{n_k+1}}\left(1 + \frac{a_{n_k+2} + 2a_{n_k+1}}{2a_{n_k+2} - a_{n_k+1}}\right)$$

and $f_4((1 + \sqrt{5})/2) = 2 + \sqrt{5}$. Since

$$\frac{5}{3} = \frac{a_4}{a_3} > \frac{a_6}{a_5} > \cdots > \frac{1 + \sqrt{5}}{2},$$

it suffices for the proof of the inequality (n_k even, $n_k \geqslant 2$)

$$\frac{a_{n_k+2}}{a_{n_k+1}}\left(1 + \frac{a_{n_k+2} + 2a_{n_k+1}}{2a_{n_k+2} - a_{n_k+1}}\right) > 2 + \sqrt{5}$$

to show, that $f_4'(x) > 0$ when $(1 + \sqrt{5})/2 \leqslant x \leqslant 5/3$. For such x we have

$$\begin{aligned} f_4'(x)(2x - 1)^2 &= 6x^2 - 6x - 1 \\ &> 6\left(\frac{1 + \sqrt{5}}{2}\right)^2 - 6 \cdot \frac{5}{3} - 1 = 3\sqrt{5} - 2 > 0. \end{aligned}$$

Thus, it remains for us to consider the expression (19) only for odd $n_k \geqslant 1$. Now, we cannot have $n_j \geqslant n_{j+1}$ for all sufficiently large j. For, we would then have $\beta \sim \alpha_m$ for some $m \geqslant 1$, i.e., $R_\beta{}^* > 2 + \sqrt{5}$ by the above remark. Hence, we must have $n_k < n_{k+1}$ for infinitely many suffixes k. Thus, let $k \geqslant 3$ and $n_k < n_{k+1}$. Then

$$\begin{aligned} R_\beta^* &> (2; c_1, c_2, ..., c_{n_k}, 2, 2) \cdot (c_1; c_2, ..., c_{n_k+1}, 3) \\ &\geqslant (2; c_1, c_2, ..., c_{n_k}, 2, 2) \cdot (c_1; c_2, ..., c_{n_k+2}, 3) \\ &= \frac{5a_{n_k+2} + 2a_{n_k+1}}{5a_{n_k} + 2a_{n_k-1}} \cdot \frac{3a_{n_k+2} + a_{n_k+1}}{3a_{n_k+1} + a_{n_k}} \\ &= \frac{5a_{n_k+2} + 2a_{n_k+1}}{3a_{n_k+2} - a_{n_k+1}} \cdot \frac{3a_{n_k+2} + a_{n_k+1}}{a_{n_k+2} + 2a_{n_k+1}}. \end{aligned}$$

Let us put

$$f_5(x) = \frac{5x + 2}{3x - 1} \frac{3x + 1}{x + 2}.$$

We have $f_5((1 + \sqrt{5})/2) = 2 + \sqrt{5}$ and

$$f_5\left(\frac{a_{n_k+2}}{a_{n_k+1}}\right) = \frac{5a_{n_k+2} + 2a_{n_k+1}}{3a_{n_k+2} - a_{n_k+1}} \cdot \frac{3a_{n_k+2} + a_{n_k+1}}{a_{n_k+2} + 2a_{n_k+1}}.$$

Since $3/2 = a_3/a_2 < a_5/a_4 < \cdots < (1 + \sqrt{5})/2$, the inequality

$$f_5\left(\frac{a_{n_k+2}}{a_{n_k+1}}\right) > 2 + \sqrt{5} \qquad (n_k \geqslant 1,\ n_k \text{ odd})$$

will be proved, when we show that $f_5'(x) < 0$ for $3/2 \leqslant x \leqslant (1 + \sqrt{5})/2$. For such x we have

$$\begin{aligned} f_5'(x)(3x - 1)^2(x + 2)^2 &= 42x^2 - 72x - 32 \\ &< 42\left(\frac{1 + \sqrt{5}}{2}\right)^2 - 72 \cdot \frac{3}{2} - 32 \\ &= 21\sqrt{5} - 77 < 0. \end{aligned}$$

Theorem 2 is proved.

COROLLARY. *The set $\mathscr{M}^*$ is not closed.*

THEOREM 3. *There exists a number R^{**} such that $[R^{**}, +\infty) \subset \mathscr{M}^*$.*

Proof. Let an arbitrary $\lambda \geqslant 83/4 + 9/\sqrt{2} = 27.11...$ be given. As was shown by Hall [4, Theorem 3.2, p. 974], λ can be written in the form $\lambda = (b_0; b_1, b_2, ...) \cdot (d_0; d_1, d_2, ...)$, where $5 \leqslant b_0 \leqslant d_0 \leqslant b_0 + 1$ and $b_j \leqslant 4$, $d_j \leqslant 4$ for $j \geqslant 1$. We construct a number $\kappa = (g_0; g_1, g_2, ...)$ as follows: $\kappa = (0; d_0, 5, d_0, b_0, b_1, b_2, 5, b_2, b_0, d_0, d_1, d_2, d_3, d_4, 5, d_4, d_3, d_2, d_1, d_0, \ldots, b_0, b_1, \ldots, b_{2i}, 5, b_{2i}, \ldots, b_1, b_0, d_0, d_1, \ldots, d_{2i+2}, 5, d_{2i+2}, \ldots, d_1, d_0, \ldots)$. We claim that $R_\kappa{}^* = \lambda$. By Lemma 1 we have $R_\kappa{}^* = \sup_{k\geqslant 1}(g_k; g_{k-1}, \ldots, g_1)(g_{k+1}, g_{k+2}, \ldots)$. Let us put

$$S_k = (g_k; g_{k-1}, \ldots, g_1) \cdot (g_{k+1}, g_{k+2}, \ldots)$$

for $k \geqslant 1$. Evidently, we have $S_1 < d_0(5; d_0) < \lambda$, $S_2 < (5; d_0)(d_0; b_0) < \lambda$. Further,

$$S_{2n^2+n} < (b_0; b_1, \ldots, b_{2n}, 5) \cdot (d_0; d_1, \ldots, d_{2n-2}, 5) < \lambda, \qquad n = 1, 3, 5, \ldots,$$
$$S_{2n^2+n} < (b_0; b_1, \ldots, b_{2n-2}, 5) \cdot (d_0; d_1, \ldots, d_{2n}, 5) < \lambda, \qquad n = 2, 4, 6, \ldots$$

while $\sup_{n\geqslant 1} S_{2n^2+n} = \lim_{n\to\infty} S_{2n^2+n} = \lambda$. We have also

$$\begin{aligned}
S_{2n^2+n+1} &< (b_0; d_0) \cdot (b_1; b_2) < \lambda, & n &= 1, 3, 5, \ldots \\
S_{2n^2+n+1} &< (d_0; b_0) \cdot (d_1; d_2) < \lambda, & n &= 2, 4, 6, \ldots \\
S_{2n^2+n-1} &< (b_0; d_0) \cdot (b_1; b_2) < \lambda, & n &= 2, 4, 6, \ldots \\
S_{2n^2+n-1} &< (d_0; b_0) \cdot (d_1; d_2) < \lambda, & n &= 3, 5, 7, \ldots \\
S_{2n^2-n+1} &< (5; b_{2n-2}) \cdot (b_{2n-2}; b_{2n-3}) < (5; 4) \cdot (4; 1) < \lambda, & n &= 2, 4, 6, \ldots \\
S_{2n^2-n+1} &< (5; d_{2n-2}) \cdot (d_{2n-2}; d_{2n-3}) < (5; 4) \cdot (4; 1) < \lambda, & n &= 3, 5, 7, \ldots \\
S_{2n^2-n} &< (5; b_{2n-2}) \cdot (b_{2n-2}; b_{2n-3}) < (5; 4) \cdot (4; 1) < \lambda, & n &= 2, 4, 6, \ldots \\
S_{2n^2-n} &< (5; d_{2n-2}) \cdot (d_{2n-2}; d_{2n-3}) < (5; 4) \cdot (4; 1) < \lambda, & n &= 3, 5, 7, \ldots
\end{aligned}$$

and, finally, $S_k < 5 \cdot 5 < \lambda$ in all remaining cases.
Hence,

$$R_\kappa{}^* = \sup_{k\geqslant 1} S_k = \sup_{n\geqslant 1} S_{2n^2+n} = \lambda.$$

Remark. The estimate $R^{**} \leqslant 83/4 + 9/\sqrt{2} = 27.11\ldots$ could be substantially improved of course, [3, Theorem 4].

References

1. J. W. S. Cassels, "An Introduction to Diophantine Approximations," Cambridge, University Press, London, 1957.
2. J. Lesca, Sur les approximations diophantiennes à une dimension, Univ. of Grenoble 1968, mimeographed.
3. B. Diviš and B. Novák, A remark on the theory of Diophantine approximations, *Comm. Math. Univ. Carolinae* **12**, 1 (1971), 127–141.
4. M. Hall, Jr., On the sum and product of continued fractions, *Ann. of Math.* **48** (1947), 966–993.
5. J. F. Koksma, "Diophantische Approximationen," Springer Verlag, Berlin und Leipzig, 1936.